Special Representations of Surface Groups

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Abstract

These notes are for my talk at the Locally Trivial Seminar on January 28, 2025. We will introduce Nonabelian Hodge Theory, especially for smooth (not necessarily proper!) curves over $\mathbb{C}$ and look at some examples of local systems.

1 Introduction

Let X be a smooth proper curve over $\mathbb{C}$, and D a collection of points on X. Let $\pi : Y \to X \setminus D$ be a smooth proper family (i.e., a smooth proper submersion) of complex varieties. Then, we can look at the derived pushforward $R^i\pi_*\underline{\mathbb{C}}$ on $X \setminus D$. This is a local system on $X \setminus D$, and corresponds to one of the "special" representations of $\pi(X \setminus D, p)$ where $p \in X \setminus D$ is a distinguished point. For some $x \in X$, let $Y_x = f^{-1}(x)$.

Given some $R^i\pi_*\underline{\mathbb{C}}$, we obtain a representation of $\pi_1(X \setminus D, p)$ by transporting classes in the fiber $(R^i\pi_*\underline{\mathbb{C}})_p \cong H^i(Y_p, \mathbb{C})$ along classes $\gamma \in \pi_1(X \setminus D, p)$. This gives us a representation $\rho_p : \pi_1(X \setminus D, p) \to \mathrm{GL}(H^i(Y_p, \mathbb{C}))$. Recall that when $X \setminus D$ is connected, the fundamental group is independent of our choice of basepoint p, up to conjugating by the choice of a homotopy class of a path. This is realized by conjugating our representation, meaning we get an isomorphic representation.

More generally, given any rank n complex local system $\mathbb{V}$, we can get an isomorphism class of an n-dimensional representation $\pi_1(X \setminus D) \to \mathrm{GL}_n(\mathbb{C})$. This operation is invertible, and hence we may view the moduli space of rank n local systems as the same as the moduli space of rank n representations of $\pi_1(X \setminus D)$. Let M_B be this moduli space of representations. Our goal is to study the points in M_B corresponding to the representations obtained from some $R^i\pi_*\underline{\mathbb{C}}$. To that end, we make the following definition

Definition 1.1. A local system $\mathbb{V}$ on $X \setminus D$ is geometric/motivic/of geometric origin if there is a Zariski dense open set $U \subseteq X \setminus D$ and a smooth proper map $f : Y \to U$ such that $\mathbb{V}|_U$ is isomorphic to a subquotient of $R^i f_*\underline{\mathbb{C}}$.

2 Hodge theory and non-abelian cohomology

It turns out that studying motivic local systems is not so easy. There are some easy necessary on a local system $\mathbb{V}$ for it to be motivic. First, we require that $\mathbb{V}$ actually is a local system defined over $\mathbb{Z}$. This is because if $\mathbb{V}$ comes from the cohomology of a complex variety, it must have an integral structure due to the existence of singular cohomology. The second is that $\mathbb{V}$ must come from a variation of Hodge structure.

A sufficient condition for motivicity is rigidity. If $\mathbb{V}$ is motivic, it is rigid. This is recent work of Joakim Faergeman and uses the Geometric Langlands Correspondence.

Definition 2.1. A variation of Hodge structure $(V, F^\bullet, \nabla)$ on $X \setminus D$ is a holomorphic flat vector bundle on $X \setminus D$ with a filtration satisfying Griffiths transversality: $\nabla(F^p) \subseteq F^{p-1} \otimes \Omega^1_X(\log D)$. A polarization is a flat Hermitian form ψ so that $(-1)^p$ is positive on F^p/F^{p+1}. A polarizable variation of Hodge structure is a variation with a polarization.

Definition 2.1 is a fake definition, but it is good enough for our purposes. The filtration $F^\bullet$ is called the Hodge filtration, and at each $x \in X \setminus D$ we get a filtration $(F^\bullet_x)$ of the fiber V_x. In this way, the Hodge filtration on ordinary cohomology $H^i(Y_x, \mathbb{C}) = \oplus_{p+q=i} H^{p,q}(X, \mathbb{C})$ is mimicked.

Conjecture 2.2. The motivic local systems are precisely the integral polarizable variation of Hodge structures.

Currently, our picture looks like this.

$$\{\text{motivic local systems}\} \subseteq \{\mathbb{Z} - \text{VHS}\} \subseteq \{\mathbb{C} - \text{VHS}\} \subseteq M_B.$$

It turns out, there are tools to study $\mathbb{C}$-VHS. For ordinary cohomology, if X is a compact Kähler manifold we know that $\text{Hom}(\pi_1(X), \mathbb{C}) = H^1_{\text{Betti}}(X, \mathbb{C}) \cong H^1_{\text{dR}}(X, \mathbb{C}) \cong H^1_{\text{Dol}}(X, \mathbb{C}) = H^0(X, \Omega^1_X) \oplus H^1(X, \mathscr{O}_X)$. That is, a singular cohomology class is the same as the data of a globally defined 1-form and gluing data for a line bundle.

We note that by definition

$$M_B := \text{Hom}(\pi_1(X), \text{GL}_n(\mathbb{C}))//\text{conjugation}.$$

In this way, M_B is a "non-abelian" Betti cohomology group for X. This is diffeomorphic to M_{Dol}, the moduli space of stable Higgs bundles on X of degree 0, which is the "non-abelian" Dolbeault cohomology group for X.

3 Parabolic Higgs Bundles

Let X be a smooth proper curve and $D = \{x_1, \dots, x_r\}$ a collection of points on X. We let $M = M_B(C_1, \dots, C_r)$ be a relative character variety where $(C_1, \dots, C_r)$ is a collection of conjugacy classes in $\text{GL}_n(\mathbb{C})$ and $M_B(C_1, \dots, C_r)$ is the space of representations ρ such that for a simple closed loop γ_i around x_i, we require that $\rho(\gamma_i) \in C_i$. The idea is that instead of studying M_B as a whole (or the $\mathbb{C}$-VHS inside it), we can study it one bit at a time. Depending on what $(C_1, \dots, C_r)$ is M may be empty or nonempty.

We restrict to the case where the the $C_1, \dots, C_r \subseteq U(n)$. This is not a real restriction because the $\mathbb{C}$-VHS must preserve a polarization and hence underly representations satisfying this unitary local monodromy data.

Definition 3.1. A parabolic Higgs bundle (E_*, θ) on a pair (X, D) (where $D = x_1 + \dots + x_r$) is a bundle E on X along with the following data:

1. a sequence of real numbers $0 \leq \alpha_i^1 < \dots < \alpha_i^{n_i+1} < 1$ at each x_i

2. a strictly decreasing flag of the fiber $E_{x_i} = E_i^1 \supset E_i^2 \supset \cdots \supset E_i^{n_i} \supset E_i^{n_i+1} = 0$.
3. A map $\theta : E \to E \otimes \Omega_X^1(\log D)$ so that at each x_i, $\theta(E_i^j) \subseteq E_i^{j+1} \otimes (\Omega_X^1(\log D))_{x_i}$.

The parabolic Higgs bundles are related to representations in $M_B(C_1, \ldots, C_r)$ in the following way: given some $A_i \in C_i$, it is diagonalizable with eigenvalues $(e^{2\pi i\theta_1}, \ldots e^{2\pi i\theta_n})$. After removing redundant θ_i's and ordering them in increasing order, we get our parabolic weights. The flags are obtained by summing the generalized eigenspaces to each of the $e^{2\pi i\theta_i}$'s.

Definition 3.2. We define the parabolic degree of a parabolic Higgs bundle to be

$$\text{par-deg}\, E_* = \deg(E) + \sum_{i=1}^{r} \sum_{j=1}^{n_i} \alpha_i^j \cdot \dim(E_i^j / E_i^{j+1})$$

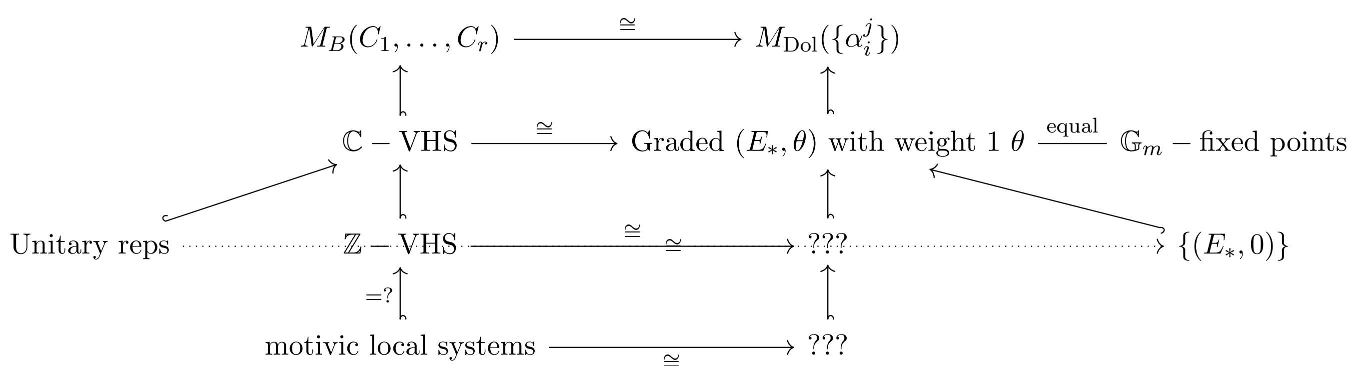
and we call the quantity $\mu_*(E_*) = \text{par-deg}(E_*)/\operatorname{rank} E$ to be the slope of E_*.

Definition 3.3. We say that a parabolic Higgs bundle (E_*, θ) is stable (resp. semistable) if for all sub-Higgs bundles (the subbundles $F \subseteq E$ satisfying $\theta(F) \subseteq F \otimes \Omega_X^1(\log D)$), we have that $\mu_*(F_*) < \mu_*(E_*)$ (resp. $\leq$).

Let $(C_1, \ldots, C_r)$ be some local monodromy data and $\{\alpha_i^j\}$ the parabolic weights associated to the data. Let $M_{\text{Dol}}(\{\alpha_i^j\})$ be the moduli space of semi-stable parabolic Higgs bundles of parabolic degree 0. Note that there is a natural $\mathbb{G}_m$-action on $M_{\text{Dol}}(\{\alpha_i^j\})$ given by scaling the Higgs field $t \cdot (E_*, \theta) \mapsto (E_*, t\theta)$.

Theorem 3.4 - (Simpson). There is a diffeomorphism $M_B(C_1, \ldots, C_r) \cong M_{\text{Dol}}(\{\alpha_i^j\})$.

Why the correspondence works (or indeed what the actual map is!) is not so important for our purposes. Instead, I will detail certain important objects on each side and how they relate to eachother.



Notably, the integral local systems and the motivic local systems do not have a good description. We are able to tell quite easily when something is a variation of Hodge structure, but there is no known current technique to tell which subring $R \subseteq \mathbb{C}$ a Higgs bundle is defined over (in the sense that the corresponding representating is defined over R). Conversely, given a representation ρ we can tell what ρ is defined over but it is not easy to see when it underlies a variation.

4 Examples

To write down a $\mathbb{C}$-VHS, we just need to write down a parabolic graded Higgs bundle with Higgs field of weight one.

Example 4.1. Let $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ be very small numbers. Then, consider the parabolic Higgs bundle with the following weights

D	$\mathscr{O}$	$\mathscr{O}(-1)$	$\mathscr{O}(-1)$	$\mathscr{O}(-1)$
x_1	$\frac{1}{4} - \epsilon_1$	$\frac{1}{4} - \frac{\epsilon_1}{2}$	$\frac{1}{4} + \frac{\epsilon_1}{2}$	$\frac{1}{4} + \epsilon_1$
x_2	$\frac{1}{4} - \epsilon_2$	$\frac{1}{4} - \frac{\epsilon_2}{2}$	$\frac{1}{4} + \frac{\epsilon_2}{2}$	$\frac{1}{4} + \epsilon_2$
x_3	$\frac{1}{4} - \epsilon_2$	$\frac{1}{4} + 2\epsilon_2$	$\frac{1}{4} - 2\epsilon_2$	$\frac{1}{4} + \epsilon_2$

Here, our Higgs field goes from left to right. By choosing our Higgs fields generically, we note that our sub-Higgs bundles are the obvious ones. Then, checking stability is just a numerical condition which is obvious because $3/4 \ll 1$.

Example 4.2. Consider the parabolic Higgs bundle

point	$\mathscr{O}(-1) \oplus \mathscr{O}(-1) = S$	$\mathscr{O}(-1) = Q$
x_1	$\frac{1}{3}, \frac{97}{300}$	$\frac{103}{300}$
x_2	$\frac{9}{20}, \frac{1}{20}$	$\frac{1}{2}$
x_3	$\frac{17}{50}, \frac{13}{20}$	$\frac{1}{100}$

where our flags are chosen generically, as is our Higgs field. The sub-Higgs bundles we need to check stability for are Q and $L \oplus \theta(L)$ for all subbundles $L \subseteq S$. Certainly $\mu_*(Q_*) < 0$. Then, any destabilizing subbundle $L \subseteq Q$ necessarily must be of degree -1. But note that because our flags were chosen generically, so if L takes the largest weight at one point it must take the smallest weights at the other two points (because the inclusion map $L \hookrightarrow S$ is constant). Then, stability is guaranteed.