BAD SIMPLEX ARGUMENTS

MATTHEW SCALAMANDRE

These are notes for an expository talk at the Locally Trivial Seminar.

1. Simplicial Complexes

We give two definitions of these objects:

Definitions 1.1.
- (Combinatorial Model): A *simplicial complex* X is a set V_X of vertices, together with a set $S_X \subseteq \mathcal{P}_{\text{fin}}(V_X)$ which contains all singletons, and is closed under subsets.
- (Topological Model): A simplicial complex is a CW-complex where D^i is modelled on Δ^i, and where the only allowed attaching maps are the inclusions $\partial\Delta^i \hookrightarrow \Delta^i$. Further, we require that every simplex is uniquely determined by its set of vertices.

These are equivalent, by taking geometric realizations or looking at vertex sets. We have the following operation:

Definition 1.2. Let X be a simplicial complex. The *barycentric subdivision* $\operatorname{sd} X$ has $V_{\operatorname{sd} X} = S_X$, where $\{\sigma_0, \dots, \sigma_n\}$ is a simplex if (in some ordering)

$$\sigma_0 \subsetneq \sigma_1 \subsetneq \cdots \subsetneq \sigma_n.$$

It is always true that X is homeomorphic to $\operatorname{sd} X$. A continuous map $f : X \to Y$ is *simplicial* if, for every simplex $\sigma \in X$, we have that $f(\sigma)$ is a simplex and $f|_\sigma$ is linear.

Requiring that maps are simplicial on the nose is quite restrictive: for example, we would like $\operatorname{Hom}(\partial\Delta^1, X)$ to be something like $\pi_1(X)$. However, we can only realize a path of length 3 by a simplicial map. To solve this, we say $f : X \to Y$ is a *map of simplicial complexes* if the induced map $f : \operatorname{sd}^k X \to \operatorname{sd}^\ell Y$ is simplicial.

Proposition 1.3. *The functor* $\boldsymbol{sCx} \to \boldsymbol{hTop}$ *is full and essentially surjective.*

Therefore, **sCx** is a model of the homotopy category. The usual model is the category of CW complexes. The category **sCx** has worse *categorical* properties, since products of simplicial complexes do *not* agree with the product in **Top**. However, the objects of **sCx** are nicer to work with geometrically.

We begin defining some useful geometric constructions:

Definition 1.4. The *join* of A and B is

$$\begin{aligned} A * B &= \text{"Draw an edge from every point of A to every point of B"} \\ &= (V_A \cup V_B, \{\sigma * \tau \mid \sigma \in S_A, \tau \in S_B\}) \\ &\cong A \times B \times [0,1]/\{(a, b_1, 0) \sim (a, b_2, 0), (a_1, b, 1) \sim (a_2, b, 1)\} \\ &\simeq \Sigma(A \wedge B) \quad \text{(pointing A,B arbitrarily).} \end{aligned}$$

Definition 1.5. Let $\sigma \in X$ be a k-simplex. Then, the *star* of σ is

$$\mathrm{st}(\sigma) = \left(\bigcup_{\sigma \leq \tau} \overline{\tau} \right) \subseteq X.$$

The *link* of σ is the complex $\mathrm{lk}(\sigma)$ whose simplex set is $\{\tau \in S_X \mid \sigma \lneq \tau\}$. (What is the vertex set?)

To relate these complexes, observe that $\mathrm{st}(\sigma) = \sigma * \mathrm{lk}(\sigma)$. Therefore, $\mathrm{st}(\sigma)$ is contractible, and $\mathrm{st}(\sigma) \setminus \mathring{\sigma} \simeq \Sigma^{\dim(\sigma)} \mathrm{lk}(\sigma)$.

Warning 1.6. $\mathrm{st}(\sigma)$ is not always a full subcomplex of X, i.e. there may be a simplex τ of X with $\partial\tau \in \mathrm{st}(\sigma)$ but $\tau \notin \mathrm{st}(\sigma)$. If this always holds, X is called a *flag complex.* Subdivisions are *always* flag complexes.

2. Bad Simplex Argument

We follow §2.1 of Hatcher–Vogtmann.

Let X be a simplicial complex, and Y be a subcomplex of X. We want to relate their homotopy types. Say that $S \subset S_X \setminus S_Y$ is a *collection of bad simplices for* X, Y if both of the following conditions hold.

1. If σ has no faces in S, then $\sigma \in Y$.
2. If two faces μ, τ of a simplex σ are bad, so is their internal join $\mu \cup \tau$.

Let $\sigma \in S$, and let

$$G_\sigma = \{\tau \in \mathrm{lk}(\sigma) \mid \text{All bad faces of } \tau \text{ lie in } \sigma\}.$$

Theorem 2.1. *If, for all $\sigma \in S$, the complex G_σ is $(j - \dim(\sigma) - 1)$-connected, then $\pi_i(X, Y) = 0$ for $i \leq j$.*

Proof. Let $i \leq j$, and let $f : (D^i, S^{i-1}) \to (X, Y)$. Without loss of generality, f is simplicial with respect to some refinement of the standard triangulation of D^i. Let μ be a maximal simplex of D^i such that $\sigma = f(\mu)$ is bad. It follows by property (2) of S that $im(f|_{\mathrm{lk}(\mu)}) \subseteq G_\sigma$. Since we are using the standard triangulation of D^i, we have

$$\mathrm{lk}(\mu) \cong S^{i-\dim(\mu)-1}.$$

Since G_σ is $(j - \dim(\sigma) - 1)$-connected and $\dim(\sigma) \leq \dim(\mu)$, the map $f|_{\mathrm{lk}(\mu)}$ is nullhomotopic in G_σ.

First, we retriangulate $\mathrm{st}(\mu) \subset D^i$ as $C\,\mathrm{lk}(\mu) * \partial\mu$. Define g to agree with f off of the interior of $\mathrm{st}(\mu)$. To define the map on the interior of $\mathrm{st}(\mu)$, hang the nullhomotopy of $f|_{\mathrm{lk}(\mu)}$ off the cone $C\,\mathrm{lk}(\mu)$, and join the result over $f|_{\partial\mu}$. Since all modifications to f happen within the contractible $G_\sigma * \sigma$, we have $f \simeq g$. We have reduced the number of simplices of D^i mapping to bad simplices. Since D^i is compact, we can repeat the process until no bad simplices are in the image. By Property 1, the image of f lies in Y, as required. □

Corollary 2.2. *Let X, Y be as above, and let j be an integer so that the conditions of Theorem 2.1 are satisfied. Then,*

$$\pi_i(Y) \cong \pi_i(X) \qquad \text{for } i < j$$

$$\pi_j(Y) \twoheadrightarrow \pi_j(X).$$

Both of these are useful results!

3. Example: Maazen

Let $\mathcal{B}_n$ be the complex whese vertices are unimodular vectors in $\mathbb{Z}^n$, and which has a k-simplex for every partial basis of size $k+1$. Let $\mathcal{B}_n^m = \mathrm{lk}(\{e_1, \dots, e_m\})$.

Theorem 3.1 (Maazen)**.** *$\mathcal{B}_n^m$ is $(n-2)$-connected.*

This implies slope-1 homological stability for $\mathrm{SL}_n(\mathbb{Z})$ (by Bernard–Miller–Sroka; Maazen found a worse range), $\mathrm{H}^{\binom{n}{2}}(\mathrm{SL}_n(\mathbb{Z}); \mathbb{Q}) = 0$ (see Church–Farb–Putman), etc.

Proof (Church–Putman): Let X_N be the full subcomplex on those vertices with $|\text{Last coordinate}| \leq N$. The complex X_0 lies in the contractible $\mathrm{st}(e_{m+n})$. Any simplicial map $S^k \to \mathcal{B}_n^m$ lies in some X_N, by compactness of S^k. If we can show that (X_N, X_{N-1}) is k-connected for all N, then $\mathcal{B}_n^m$ is k-connected.

We show this by a bad simplex argument. Let S be the set of simplices all of whose vertices have $|\text{last coordinate}| = N$. If σ is such a simplex, then

$$G_\sigma = \mathrm{lk}^{<N}(\sigma) = \mathrm{lk}(\sigma) \cap X_{N-1}.$$

We need to show that G_σ is $(n - \dim(\sigma) - 3)$-connected. The full link is highly enough connected, by induction, since

$$\mathrm{lk}(\sigma) \cong \mathcal{B}_{n-\dim(\sigma)-1}^{m+\dim(\sigma)+1}.$$

To show that G_σ carries the same connectivity, we construct a retraction:

Lemma 3.2. *Let $\sigma \in S$, and choose $w \in \sigma$ with last coordinate $\pm N$. There is a retraction $\rho : \mathrm{lk}(\sigma) \to G_\sigma$, defined on vertices by*

$$\rho(v) = v \mp \left\lfloor \frac{v_{m+n}}{N} \right\rfloor w.$$

Exercise 3.3. Prove Lemma 3.2 by verifying that ρ extends over the simplices of $\mathrm{lk}(\sigma)$.

Returning to the proof of Maazen's theorem, if $i \leq (n - \dim(\sigma) - 3)$, we have:

$$\begin{array}{ccccc} & & 0 & & \\ & & \| & & \\ \pi_i(G_\sigma) & \xrightarrow{\iota_*} & \pi_i(\mathrm{lk}(\sigma)) & \xrightarrow{\rho_*} & \pi_i(G_\sigma). \end{array} \qquad \rho_* \circ \iota_* = \mathrm{id}_{G_\sigma}$$

It follows that G_σ is $(n - \dim(\sigma) - 3)$-connected, and therefore that (X_N, X_{N-1}) is $(n-2)$-connected. Therefore, $\mathcal{B}_n^m$ is $(n-2)$-connected. □