LOCAL SYSTEMS IN SIMPLE HOMOTOPY THEORY

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Contents

1. Introduction

This talk will be about simple homotopy theory, the field that tries to answer the question: *To what extent can homotopy theory be reduced to the geometry/combinatorics of finite CW-complexes.* Our goal is to motivate a modern perspective on some classical objects of study in simple homotopy theory.

As a standing assumption, all spaces will be connected.

2. Wall's Obstruction

There are two very natural questions one could ask. A classical question:

Question 2.1. *If Y is a finite CW-complex, and $r\colon Y \to X$ is a retract, i.e. there is a map $s\colon X \to Y$ such that $r \circ s \simeq \mathrm{id}_X$, is X also a finite CW-complex? We call such an X finitely dominated.*

And a modern question:

Question 2.2. *Is the map $\mathcal{S}^{\mathrm{fin}} \to \mathcal{S}^{\omega}$ (where ω denotes compact objects) an equivalence?*

These are the same question. Being a compact object in spaces is the same as being homotopy equivalent to a finitely dominated CW-complex. These questions were answered in the negative by Wall, in the following way.

For a finitely dominated space X let $\widetilde{X} \to X$ be the universal cover of X. Then Wall associates to $C_\bullet(\widetilde{X})$ an element of $\widetilde{K}_0(\mathbb{Z}[\pi_1(X)])$ called $w(C_\bullet(\widetilde{X}))$ or just $w(X)$. This process of taking $w(X)$ should be thought of as "taking an alternating sum" and relies heavily on X being finitely dominated.

Theorem 2.3 (Wall [Wal65])**.** *Suppose that X is finitely dominated. Then X is finite if and only if $w(X) = 0$. Further for element of $w \in \widetilde{K}_0(\mathbb{Z}[G])$ there is a finitely dominated space X with $\pi_1(X) = G$ and $w(X) = w$.*

This w is called the *Wall finiteness obstruction.* The modern construction of this obstruction will look somewhat different. To begin, we modify where the obstruction will live.

$$K(\mathbb{Z}[\pi_1(X)]) \simeq_{\pi_0} K(H\mathbb{Z}[\Omega X]) \simeq_{\pi_0} K(\mathbb{S}[\Omega X]) \simeq K(\mathrm{Mod}^{\omega}_{\mathbb{S}[\Omega X]}).$$

Then using Schwede-Shipley [Lur17] 7.1.2.3 we know that $\mathrm{Mod}_{\mathbb{S}[\Omega X]} \simeq \mathrm{Fun}(X, \mathcal{S}\mathrm{p})$, this $\mathrm{Fun}(X, \mathcal{S}\mathrm{p})$ or $\mathcal{S}\mathrm{p}^X$ is what we call the category of local systems of spectra on X.

In the remainder of this talk, we will develop simple homotopy in $K((\mathcal{S}\mathrm{p}^X)^\omega)$.

3. Local Systems.

The first notable thing about $\mathcal{S}\mathrm{p}^X$ is that it has a six-functor formalism. We will not describe the entirety of this formalism here, but some of the functors will play a roll. It is worth remarking that the six-functor formalism for local systems is especially good, for example every pushout square of spaces gives a Beck-Chevalley (base change) square.

Definition 3.1. Let $\mathbb{S}_X$ denote the constant local system with value $\mathbb{S}$ on X. Then $w(X) = \widetilde{[\mathbb{S}_X]}$.

Before we proceed we need to make sure that $\mathbb{S}_X$ is actually a compact object in $\mathcal{S}\mathrm{p}^X$. Let $\pi : X \to x$, then $\mathbb{S}_X = \pi^*\mathbb{S}$ where the upper star denotes the pullback for functor categories. So we want that π^* preserves compact objects. This is true by abstract nonsense whenever $\mathrm{fib}(\pi) \simeq X$ is compact in $\mathcal{S}$.

We will now prove a Mayer-Vietoris type result for $\mathbb{S}_X$. For this week need some version of functoriality of $\mathcal{S}\mathrm{p}^{[-]}$. The most natural choice is the pullback along a functor, but as we know this does not always preserve compact objects. This would also make $\mathcal{S}\mathrm{p}^{[-]}$ contravariant, but we want it to be covariant (the functoriality of the group ring functor is covariant). For this reason, to a functor $f : X \to Y$ we assign the functor $f_! : \mathcal{S}\mathrm{p}^X \to \mathcal{S}\mathrm{p}^Y$, the left adjoint to f^* given by left Kan extension. This $f_!$ *always* preserves compact objects.

Theorem 3.2. *Suppose that we have a pushout in* $\mathcal{S}$

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & D \end{array} .$$

Let i_A*,* i_B*, and* i_C *denote the maps from* A*,* B*, and* C *into* D*. Then the sequence*

$$(i_A)_!\mathbb{S}_A \to (i_B)_!\mathbb{S}_B \oplus (i_C)_!\mathbb{S}_C \to \mathbb{S}_D$$

is exact in $\mathcal{S}\mathrm{p}^D$*.*

Proof. The pushout gives a pushout in the over category $\mathcal{S}_{/X}$. Lurie [Lur09] 3.2.0.1 shows that straightening map $\mathcal{S}_{/X} \to \mathrm{Fun}(X, \mathcal{S})$ given by $(Y \xrightarrow{f} X) \mapsto f_! *_Y$ is an equivalence. This equivalence takes our pushout in $\mathcal{S}_{/D}$ to the pushout

$$(i_A)_! *_A \to (i_B)_! *_B + (i_C)_! *_C \to *_D$$

in $\mathrm{Fun}(D, \mathcal{S})$. Then by applying the functor $\Sigma^\infty_+ : \mathrm{Fun}(D, \mathcal{S}) \to \mathrm{Fun}(D, \mathcal{S}\mathrm{p})$ we get exactly the sequence we desire. □

Before we move on to Wall's theorem we need one last thing.

Definition/Theorem 3.3. Let $X \in \mathcal{S}^\omega$, let $\pi_X : X \to *$ be the terminal map. Then define $\chi_X = (\pi_X)_!(\mathbb{S}_X)$, then $[\chi_X]$ is the Euler characteristic of X.

Proof. To check this we can verify this on $\emptyset$, $*$, and the check on pushouts of spaces by applying the Sum theorem for K-theory to the exact sequence given by the theorem above. □

4. A Partial Proof of Wall's Theorem

Proof of Theorem 2.3. Suppose that $X \in \mathcal{S}^{\mathrm{fin}}$. We want to show that $[\mathcal{S}\mathrm{p}^X]$ vanishes in the cofibre of the map $x_! : K((\mathcal{S}\mathrm{p}^*)^\omega) \to K((\mathcal{S}\mathrm{p}^X)^\omega)$ where $x : * \to X$. To do this we can lift $[\mathbb{S}_X]$ along $x_!$. We will show that $[\chi(X)]$ is a lift for $[\mathbb{S}_X]$. Since X is finite it can be built by pushing out with a point finitely many times, i.e. it can be built from finitely many cells. We will "induct on cells".

To start, we remark that this is trivially true for a point.

Now suppose that we have a push out

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & \ulcorner & \downarrow \\ B & \longrightarrow & D \end{array}$$

where we know that the Euler characteristic lifts the constant local system for A, B, and C. Let a, b, c, and d be basepoints for each space. We have that

$$[d_! \chi(B)] = [(i_B)_! b_! \chi(B)] = [(i_B)_! \mathbb{S}_B]$$

Then we know that

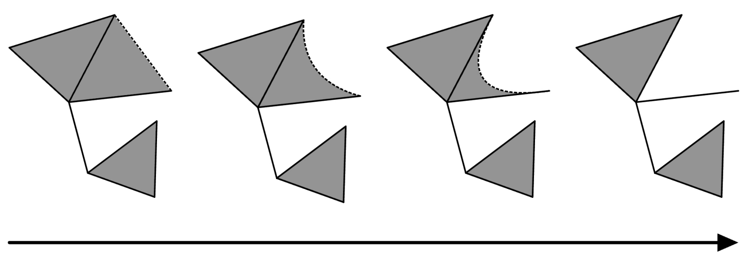
$$d_![\chi(D)] = [d_!\chi(B)] + [d_!\chi(C)] - [d_!\chi(A)]) = [(i_B)_!\mathbb{S}_B] + [(i_C)_!\mathbb{S}_C] - [(i_A)_!\mathbb{S}_A] = \mathbb{S}_D.$$

□

The ease of these proofs did not depend on working in local systems of spectra, it came from working in the language of ∞-categories. There are cases where we can use the power of the six-functor formalism for local systems to prove things about Wall's obstruction, but we derive our motivation for working in local systems from a different perspective.

5. Whitehead's Torsion

Definition 5.1. If X is a finite simplicial complex then a *simple collapse* of X looks like:



This yields a homotopy equivalent space X'. A *simple expansion* is the inverse.

Question 5.2. *Is every homotopy equivalence $f : Y \to X$ homotopy equivalent to a finite number of simple expansions and collapses?*

If f is homotopic to a finite number of simple expansions and collapses then f is called *simple*. This question was also answered in the negative by Whitehead. This time Whitehead assigned to f an element $\tau(f)$ in $\mathrm{Wh}(\pi_1(X))$ (some quotient of $K_1(\mathbb{Z}[\pi_1(X)])$). Whitehead then produces a theorem very similar to Wall[1].

Theorem 5.3 (Whitehead [Whi50])**.** *Suppose that $f : Y \to X$ is a map finite simplicial complexes. Then f is simple if and only if $\tau(f) = 0$. Further for element of $\tau \in \mathrm{Wh}(G)$ there is an equivalence $f : Z \to X$ with $\tau(f) = \tau$.*

[1] Although Whitehead was actually first.

We detail how $\tau(f)$ can be seen from the local systems perspective. It is worth remarking that the torsion of f depends explicitly on the simplicial structures for X and Y.

For the map which allowed us to understand Wall's obstruction

$$K(\mathcal{S}\mathrm{p}^*) \to K(\mathcal{S}\mathrm{p}^X)$$

to also allow us to understand Whitehead's torsion we need to upgrade to a map

$$K(\mathcal{S}\mathrm{p}^*) \otimes \Sigma^\infty_+ X \to K(\mathcal{S}\mathrm{p}^X).$$

This should've been the map we were using all along if we wanted to do Wall's obstruction for non-simply connected spaces. We give a brief definition of this map.

Definition 5.4. Define $A(X) = K(\mathcal{S}\mathrm{p}^X)$, this is the *A-theory* of X. Writing $X \simeq \mathrm{colim}_X *$ we get a map

$$\underset{X}{\mathrm{colim}}\, A(*) \to A(\underset{X}{\mathrm{colim}}\, *)$$

or in other words, a map

$$\alpha_X : A(*) \otimes \Sigma^\infty_+ X \to A(X).$$

This $\alpha_{[-]}$ is a natural transformation of functors called *assembly*. We call the cofibre of assembly $\mathrm{Wh}(X)$ the *Whitehead spectrum* of X.

So for a space X Wall's obstruction is a point in the fibre. This allows us to define our analogue of a finite simplical structure.

Definition 5.5. A *finiteness structure* on X is a lift of the point $\mathbb{S}_X$ in $A(X)$ along the assembly map, i.e. a point $\ell_X \in A(*) \otimes \Sigma^\infty_+ X$ and a path $\varphi_X : \alpha_X(\ell_X) \to \mathbb{S}_X$.

This finally allows us to define the Whitehead torsion.

Definition 5.6. Suppose $f\colon X' \to X$ is an equivalence, and X and X' are given finiteness structures. Let $\varepsilon_f\colon f_! f^* \to \mathrm{id}$ be the counit of the adjunction $f_! \dashv f^*$. Then the composition

$$\alpha_X(f(\ell_{X'})) \xrightarrow[f_!(\varphi_{X'})]{} f_!\mathbb{S}_Y \xrightarrow[\varepsilon_f]{} \mathbb{S}_X \xrightarrow[\varphi_X^{-1}]{} \alpha_X(\ell_X)$$

gives a loop in $\mathrm{Wh}(X)$ which we call τ_f the *torsion* of f.

References

[Lur09] J. Lurie, *Higher topos theory*, *Annals of Mathematics Studies*, volume 170, Princeton University Press, Princeton, NJ, 2009. 2

[Lur17] ——, *Higher algebra*, 2017, unpublished. Available online at https://www.math.ias.edu/~lurie/. 2

[Wal65] C. T. C. Wall, *Finiteness conditions for cw-complexes*, Annals of Mathematics, **81** (1965) (1), 56–69. 1

[Whi50] J. H. C. Whitehead, *Simple homotopy types*, American Journal of Mathematics, **72** (1950) (1), 1–57. 3